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FOURIER TRANSFORM FOR QUANTUM D-MODULES VIA THE PUNCTURED TORUS MAPPING CLASS GROUP

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Abstract. We construct a certain cross product of two copies of the braided dual $\tilde{H}$ of a quasitriangular Hopf algebra H, which we call the elliptic double E_H, and which we use to construct representations of the punctured elliptic braid group extending the well-known representations of the planar braid group attached to H. We show that the elliptic double is the universal source of such representations. We recover the representations of the punctured torus braid group obtained in [Jo], and hence construct a homomorphism to the Heisenberg double D_H, which is an isomorphism if H is factorizable.

The universal property of E_H endows it with an action by algebra automorphisms of the mapping class group $\widetilde{SL_2(\mathbb{Z})}$ of the punctured torus. One such automorphism we call the quantum Fourier transform; we show that when $H = U_q(\mathfrak{g})$, the quantum Fourier transform degenerates to the classical Fourier transform on $D(\mathfrak{g})$ as $q \to 1$.

1. Introduction

Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra, and let $\tilde{H}$ denote the braided dual – also known as the reflection equation algebra – of H [DKM, DM2, DM1, Ma]. This is the restricted dual vector space H°, but the multiplication is twisted from the standard one by the R-matrix (see Section 2 for details).

Let $\{e_i\}$ and $\{e^i\}$ denote dual bases of H and $\tilde{H}$, respectively. Then the canonical element $X = \sum e^i \otimes e_i \in \tilde{H} \otimes H$ is known to satisfy the following relation in $\tilde{H} \otimes H^{\otimes 2}$:

$$X^{0,12} := (\mathrm{id} \otimes \Delta)(X) = (\mathcal{R}^{1,2})^{-1} X^{0,2} \mathcal{R}^{1,2} X^{0,1} \tag{1.1}$$

Here, $\tilde{H}$ has index 0 in the tensor product, and Δ denotes the coproduct of H.

There is a canonical action of the planar braid group $B_n(\mathbb{R}^2)$ on the nth tensor $V^{\otimes n}$ power of any H-module V. Given modules M for $\tilde{H}$ and V for H, equation (1.1) allows one to define a similarly canonical action of the punctured planar braid group $B_n(\mathbb{R}^2 \backslash disc)$ on $M \otimes V^{\otimes n}$, and moreover to show that $\tilde{H}$ is universal for this action. We have:

Theorem 1.1 ([DKM], Prop 10)**.** *Let B be an algebra, and suppose that $X_B \in B \otimes H$ satisfies relation (1.1). Then there is a unique homomorphism $\phi_B : \tilde{H} \to B$ such that $(\phi_B \otimes \mathrm{id})(X) = X_B$.*

The main goal of this paper is to define elliptic analogs of the reflection equation algebra. The punctured elliptic braid group $B_n(T^2 \backslash disc)$ is the free product of two copies of $B_n(\mathbb{R}^2 \backslash disc)$, modulo certain relations. In Section 3 we construct an algebra E_H as a certain crossed product of two copies of $\tilde{H}$, mimicking the cross relations of $B_n(T^2 \backslash disc)$. We define canonical elements $X, Y \in E_H \otimes H$ by

$$X = \sum (e^i \otimes 1) \otimes e_i, \qquad Y = \sum (1 \otimes e^i) \otimes e_i,$$

and characterize the cross relations on E_H as follows:

Theorem 1.2. *The cross relations of E_H are equivalent to the following commutation relation for $X, Y, \mathcal{R}$:*

$$X^{0,1}\mathcal{R}^{2,1}Y^{0,2} = \mathcal{R}^{2,1}Y^{0,2}\mathcal{R}^{1,2}X^{0,1}\mathcal{R}^{2,1} \tag{1.2}$$

We prove the following elliptic analog of Theorem 1.1:

Theorem 1.3. *Let B be an algebra, and $X_B, Y_B \in B \otimes H$ satisfying* (1.1) *individually, and* (1.2) *together. Then there exists a unique algebra morphism*

$$\phi_B : E_H \longrightarrow B$$

such that $X_B = (\phi_B \otimes \mathrm{id})(X)$ and $Y_B = (\phi_B \otimes \mathrm{id})(Y)$. Explicitly, ϕ_B is given by

$$\phi_B(x \otimes 1) = (\mathrm{id} \otimes x)(X_B) \qquad \phi_B(1 \otimes x) = (\mathrm{id} \otimes x)(Y_B).$$

Equation (1.2) can be used to define representations of $B_n(T^2\backslash disc)$ in the same way as (1.1) is used for $B_n(\mathbb{R}^2\backslash disc)$; see Theorem 4.3. Recall that $B_n(T^2\backslash disc)$ carries a natural action of the punctured torus mapping class group, which is isomorphic to a certain central extension $\widetilde{SL_2(\mathbb{Z})}$ of $SL_2(\mathbb{Z})$. In the case H is a *ribbon* Hopf algebra, we show that this extends to an action of $\widetilde{SL_2(\mathbb{Z})}$ on E_H.

When $H = U_q(\mathfrak{g})$, we produce degenerations of E_H to the algebras of differential operators on G and, upon further degeneration, on $\mathfrak{g}$. Recall that the algebra of differential operators on an algebraic group G can be constructed as a semi-direct product

$$D(G) = U(\mathfrak{g}) \ltimes O(G),$$

where the action of $U(\mathfrak{g})$ on $O(G)$ is induced by that of $\mathfrak{g}$ on G by left invariant differential operators. This construction can be extended to any Hopf algebra and is known as the Heisenberg double [STS]. This is a semi-direct product $D_H = H \ltimes H^\circ$, where H acts on its dual by the right coregular action.

In [Jo], canonical functors are constructed from the category of modules over the Heisenberg double of a quasi-triangular Hopf algebra to the category of modules over the (unpunctured) torus braid group. This relies upon an alternate construction – due to Varagnolo-Vasserot [VV] – of the Heisenberg double of a quasi-triangular Hopf algebra, which uses the braided dual $\tilde{H}$ in place of H°. This presentation for the Heisenberg double also yields an isomorphism with the handle algebras $S_{1,1}$ of [AGS] (see Remark 3.4).

Lifting the constructions of [Jo] to the unpunctured torus braid group, they can easily be re-interpreted as producing canonical elements X and Y in $D_H \otimes H$, satisfying equations (1.1) and (1.2). Hence, Theorem 1.3 yields a unique homomorphism $\Phi : E_H \to D_H$, compatible with the representations of the $B_n(T^2\backslash disc)$ on both sides. The map Φ is an isomorphism if, and only if, H is *factorizable*. Since the quantum group $U_q(\mathfrak{g})$ is factorizable, we may identify the elliptic double $E_{U_q(\mathfrak{g})}$ with the algebra $D_q(G) := D_{U_q(\mathfrak{g})}$ of quantum differential operators on G.

In particular we obtain an $\widetilde{SL_2(\mathbb{Z})}$ action on $D_q(G)$ by the above considerations. One such automorphism of $D_q(G)$ we call the *quantum Fourier transform*; its classical limit upon an appropriate degeneration is the classical Fourier transform on the Weyl algebra $D(\mathfrak{g})$. We expect that our quantum Fourier transform for $D_q(G)$ will be compatible with that on the braided dual of $U_q(\mathfrak{g})$ defined in [LM], realizing the braided dual as an $\widetilde{SL_2(\mathbb{Z})}$-equivariant $D_q(G)$-module. Studying this category of $\widetilde{SL_2(\mathbb{Z})}$-equivariant $D_q(G)$-modules more generally is an interesting direction of future research.

Acknowledgments. This paper is a companion to work in progress with D. Ben-Zvi [BZBJ], in which we generalize the elliptic double construction to arbitrary genus, and to any braided tensor category, using the language of topological field theory. We are grateful to D. Ben-Zvi, and to all three authors of [CEE], for their many helpful discussions and encouragement, and to P. Roche for bringing the article [AGS] to our attention.

2. The braided dual and its relatives

Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra, and denote by:

- $H^e = H^{coop} \otimes H$ where H^{coop} is H with opposite comultiplication
- $H^{[2]}$ the Hopf algebra which is $H \otimes H$ as an algebra, and with coproduct given by

$$\tilde{\Delta}(x \otimes y) = (\mathcal{R}^{2,3})^{-1}(\tau^{2,3} \circ \Delta(x \otimes y))\mathcal{R}^{2,3}$$

where $\tau(a \otimes b) = b \otimes a$. Recall that the twist H^F of H by an invertible element $F \in H \otimes H$ is the Hopf algebra with the same multiplication, and with coproduct given by

$$\Delta^F(x) = F^{-1}\Delta(x)F.$$

In order for H^F to be co-associative, F must satisfy two conditions:

$$F^{12,3}F^{1,2} = F^{1,23}F^{2,3}, \qquad (\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1.$$

Two twists F, F' are *equivalent* if there exists an invertible element $x \in H$, such that $\epsilon(x) = 1$ and

$$F' = \Delta(x)F(x^{-1} \otimes x^{-1}).$$

The following is standard (see [Dr2]):

Proposition 2.1. *A twist induces a tensor equivalence H-mod $\to$ H^F-mod. Equivalent twists leads to isomorphic tensor functors.*

It is easily checked that $F = \mathcal{R}^{1,3}\mathcal{R}^{1,4} \in (H^e)^{\otimes 2}$ is a twist, and that

$$H^{[2],coop} = (H^e)^F.$$

Let D be the "double braiding" $\mathcal{R}^{2,1}\mathcal{R}^{1,2}$. Since $D\Delta(x) = \Delta(x)D$ for all x, we have:

$$H^D = H$$

as Hopf algebras. Similarly, $H^{[2],coop}$ is in fact equal to $(H^e)^{F(D^{1,3})^k}$ for any $k \in \mathbb{Z}$, with F as above.

Let H° be the restricted Hopf algebra dual of H. It has a natural H-bimodule structure, hence a H^e left module structure given by:

$$(x \otimes y) \triangleright f := f(S^{-1}(x) \cdot y)$$

where S is the antipode of H and we use the fact that S^{-1} is a Hopf algebra isomorphism $H^{coop} \to H_{op}$. It turns H° into an algebra in H^e-mod.

Remark 2.2. We use the inverse of the antipode rather than the antipode itself because it is convenient to consider the canonical element as an invariant element of of $H^\circ \otimes H$, the image of $1 \in \mathbb{C}$ under the evaluation map $\mathbf{k} \to H^\circ \otimes H$, which means that H° really denotes the *left* dual of H in the rigid monoidal category of H-modules. This is slightly different from the convention used in [DKM, Jo] but it allows us to label tensor factors from left to right.

Definition 2.3. *The kth twisted braided dual $\tilde{H}_k$ is the algebra image of H° via the tensor functor H^e-mod $\to H^{[2],coop}$-mod given by the twist $F(D^{1,3})^k$. Explicitly, this is H° as a vector space, with multiplication given by*

$$x \cdot y = m(\mathcal{R}^{1,3}\mathcal{R}^{1,4}(D^{1,3})^k \triangleright (x \otimes y))$$

where m is the multiplication of H°. This is an algebra in the category of $H^{[2],coop}$-module with the same action as above, namely

$$(x \otimes y) \triangleright f = (u \mapsto f(S^{-1}(x)uy)).$$

Let X be the canonical element of $\tilde{H} \otimes H$, that is the image of 1 under the coevaluation map $\mathbf{k} \to \tilde{H} \otimes H$. If e_i is a basis of H and e^i the dual basis of $\tilde{H} \cong H^\circ$, then $X = \sum e^i \otimes e_i$. If H is infinite dimensional then X lives in an appropriate completion of the tensor product.

Proposition 2.4. *The element X satisfies:*

$$X^{0,12} = D^k(\mathcal{R}^{1,2})^{-1}X^{0,2}\mathcal{R}^{1,2}X^{0,1}. \tag{2.1}$$

This implies that X satisfies the reflection equation

$$\mathcal{R}^{2,1}X^{0,2}\mathcal{R}^{1,2}X^{0,1} = X^{0,1}\mathcal{R}^{2,1}X^{0,2}\mathcal{R}^{1,2}.$$

The braided dual is in fact universal for this property in the following sense:

Proposition 2.5. *Let B be an algebra and $X_B \in B \otimes H$ satisfying equation (2.1) for some $k \in \mathbb{Z}$. Then there exists a unique algebra morphism*

$$\phi_B : \tilde{H}_k \longrightarrow B$$

such that $(\phi_B \otimes \mathrm{id})(X) = X_B$. Explicitly, ϕ_B is given by

$$H^\circ \cong \tilde{H} \ni f \longmapsto (f \otimes id)(X).$$

Propositions 2.4 and 2.5 are proved in [DKM] in the case $k = 0$. The general proof is similar. Note that the fact that these axioms all leads to the same reflection equation, regardless of the value of k, essentially follows from the fact that the left hand side of (2.1) is invariant under conjugation by D.

Let $u = m((S \otimes \mathrm{id})(R^{2,1}))$ where m is the multiplication of H. Then $\nu = uS(u)$ is central and satisfies

$$\Delta(\nu) = D^{-2}(\nu \otimes \nu)$$

implying that

$$D^{k-2} = \Delta(\nu)D^k(\nu^{-1} \otimes \nu^{-1})$$

meaning that D^{k-2} and D^k are equivalent. Therefore, they lead to isomorphic tensor functors, from which follows the following:

Proposition 2.6. *For any $k \in \mathbb{Z}$, the algebras $\tilde{H}_k$ and $\tilde{H}_{k+2}$ are isomorphic.*

Therefore, it is enough to consider $\tilde{H}_0$ and $\tilde{H}_1$. Moreover, if H is a ribbon Hopf algebra, then by definition ν admits a central square root implying by a similar argument:

Proposition 2.7. *If H is a ribbon Hopf algebra then all the $\tilde{H}_k$ are isomorphic.*

Remark 2.8. The algebra $\tilde{H}_0$ is usually called the reflection dual, the braided dual or the reflection equation algebra in the literature.

Remark 2.9. For any k, equation (2.1) plays the same role in the reflection equation, as the hexagon axiom in the Yang-Baxter equation, encoding some kind of compatibility with the tensor product of H-modules. Topologically, it corresponds to a "strand doubling" operation for the additional generator of the braid group of the punctured plane. Formally, such an operation depends on the choice of a framing, while a ribbon element removes the dependence on the framing.

3. The elliptic double

Let T denote the following element in $(H^{[2],coop})^{\otimes 2}$, which we identify as a vector space with $H^{\otimes 4}$:

$$T = (\mathcal{R}^{3,2})^{-1}(\mathcal{R}^{3,1})^{-1}(\mathcal{R}^{4,2})^{-1}\mathcal{R}^{1,4}.$$

Proposition 3.1. *The element T satisfies the hexagon axioms*

$$(\mathrm{id}\otimes\Delta_{H^{[2],coop}})T = T^{1,3}T^{1,2} \qquad (\Delta_{H^{[2],coop}}\otimes\mathrm{id})T = T^{1,3}T^{2,3}$$

in $(H^{[2],coop})^{\otimes 3}$.

Proof. This is straightforward computation with the Yang-Baxter equation. The computation is depicted in braids in Figure 1.

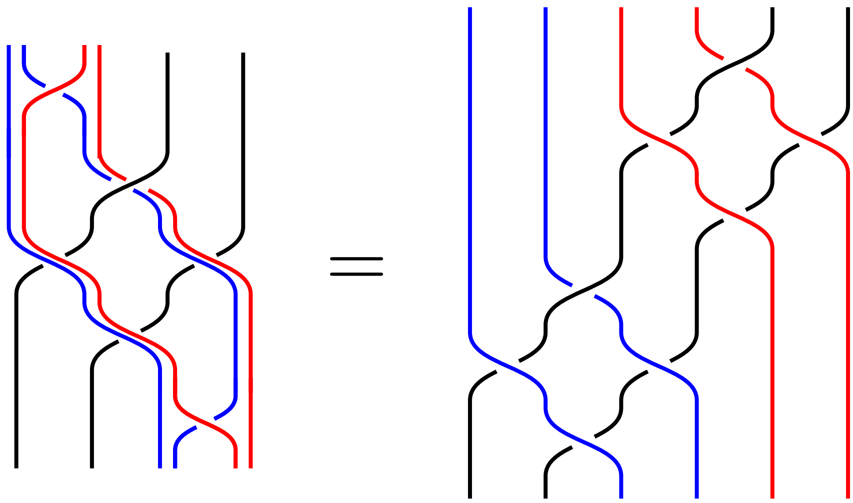

Figure 1. A braid diagram proof of $(\mathrm{id}\otimes\Delta)(T) = T_{1,3}T_{1,2}$.

□

Since $\tilde{H}_k$ is a $H^{[2],coop}$-module algebra, one can make the following definition:

Definition 3.2. *The kth* elliptic double *$E_H^{(k)}$ of H is the braided tensor square of $\tilde{H}_k$ with respect to T. Explicitly, it is $\tilde{H}_k^{\otimes 2}$ as a vector space, $\tilde{H}_k\otimes 1$ and $1\otimes\tilde{H}_k$ are subalgebras and the cross relations are given by*

$$(1\otimes g)(f\otimes 1) = T \triangleright (f\otimes g).$$

The fact that $E_H^{(k)}$ is indeed an associative algebra follows from the hexagon axioms. Choose a basis $(e_i)_{i\in I}$ of H and define $X, Y \in E_H^{(k)}\otimes H$ by

$$X = \sum e^i\otimes 1\otimes e_i, \qquad Y = \sum 1\otimes e^i\otimes e_i,$$

where we use the vector space identification $E_H^{(k)} \cong \tilde{H}^{\otimes 2}$. The main result of this section is the following:

Theorem 3.3. *The cross relations of E_H are equivalent to the commutation relation for $X, Y, \mathcal{R}$:*

$$X^{0,1}\mathcal{R}^{2,1}Y^{0,2} = \mathcal{R}^{2,1}Y^{0,2}\mathcal{R}^{1,2}X^{0,1}\mathcal{R}^{2,1}. \tag{3.1}$$

Proof. By definition every element $f \in \tilde{H}_k$ can be written as

$$f = \sum e^i f(e_i)$$

hence the product gf in $E_H^{(k)}$ is obtained by applying $(\mathrm{id}_{E_H^{(k)}} \otimes f \otimes g)$ to

$$Y^{0,2} X^{0,1}$$

and fg by applying the same element to

$$X^{0,1} Y^{0,2}.$$

Therefore all commutations relation can be gathered into a "matrix" equation

$$Y^{0,2} X^{0,1} = T \rhd_0 X^{0,1} Y^{0,2} \tag{3.2}$$

where T acts on the $E_H^{(k)}$ (i.e. 0th) component. We recall the following identities:

$$\mathcal{R}^{-1} = (S \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes S^{-1})(\mathcal{R}). \tag{3.3}$$

Applying S^{-1} to the first factor of the relation $(S \otimes \mathrm{id})(R)R = 1$, setting $\mathcal{R} = \sum r_1 \otimes r_2 = \sum r_1' \otimes r_2'$ – using apostrophes to distinguish between copies of $\mathcal{R}$ – one has the following useful identity (note the order of the terms):

$$\sum S^{-1}(r_1) r_1' \otimes r_2' r_2 = 1. \tag{3.4}$$

Then equation (3.2) reads, in coordinates:

$$\begin{aligned}&\big((1 \otimes e^j)(e^i \otimes 1)\big) \otimes e_i \otimes e_j \\ &\qquad = \big((r_2 r_1' \otimes r_2'''' r_2'' \otimes S(r_1'''') S(r_1) \otimes S(r_1'') r_2') \rhd e^i \otimes e^j\big) \otimes e_i \otimes e_j.\end{aligned} \tag{3.5}$$

The left $H^{[2]}$ action on $\tilde{H}_k$ is by definition dual to the right $H^{[2]}$ action on H, therefore:

$$\sum ((x \otimes y) \rhd e^i) \otimes e_i = \sum e^i \otimes S^{-1}(x) e_i y$$

Using this, equation (3.5) can be rewritten

$$\big((1 \otimes e^j)(e^i \otimes 1)\big) \otimes e_i \otimes e_j = e^i \otimes e^j \otimes S^{-1}(r_1') S^{-1}(r_2) e_i r_2'''' r_2'' \otimes r_1 r_1'''' e_j S(r_1'') r_2'.$$

Then, using the R-matrix relations (3.3) and (3.4) to move elements from the right hand side to the left hand side (and reassigning apostrophes for the sake of clarity) we obtain:

$$\big((1 \otimes e^j)(e^i \otimes 1)\big) \otimes r_2 r_1' e_i r_2'' \otimes r_1 e_j r_2' r_1'' = e^i \otimes e^j \otimes e_i r_2 \otimes r_1 e_j$$

which is exactly (1.2). □

Remark 3.4. The relations of Theorem 3.3 should be compared with those of the graph algebra $S_{1,1}$ of [AGS].

Equation (1.2) is a defining relation for $E_H^{(k)}$, in the following sense:

Corollary 3.5. *Let B be an algebra, and $X_B, Y_B \in B \otimes H$ satisfying both the axiom* (2.1) *and equation* (1.2) *(with X and Y replaced by X_B and Y_B). Then there exists a unique algebra morphism*

$$\phi_B : E_H^{(k)} \longrightarrow B$$

such that $X_B = (\phi_B \otimes \mathrm{id})(X)$ and $Y_B = (\phi_B \otimes \mathrm{id})(Y)$. Explicitly, ϕ_B is given by

$$\phi_B(x \otimes 1) = (\mathrm{id} \otimes x)(X_B) \qquad \phi_B(1 \otimes x) = (\mathrm{id} \otimes x)(Y_B).$$

4. Braid group and mapping class group actions

In this section we construct representations of the punctured torus braid group from $E_H^{(k)}$. First, we have:

Definition 4.1. *The punctured elliptic braid group $B_n(T^2\backslash disc)$ is the fundamental group of the configuration space of n points in $T^2\backslash disc$.*

Proposition 4.2. *The group $B_n(T^2\backslash disc)$ is generated by $X_1, \dots, X_n, Y_1, \dots, Y_n, \sigma_1, \dots, \sigma_{n-1}$ with relations:*
- *the X_i's (resp. $Y_i' s$) pairwise commute,*
- *the planar braid relation for the σ_i's,*
- *the following cross relations:*

$$X_{i+1} = \sigma_i X_i \sigma_i \qquad\qquad Y_{i+1} = \sigma_i Y_i \sigma_i \tag{4.1}$$

$$X_1 Y_2 = Y_2 X_1 \sigma_1^2 \tag{4.2}$$

The results of the previous section easily imply:

Theorem 4.3. *There exists unique group morphisms*

$$\phi : B_n(T^2\backslash disc) \longrightarrow (E_H^{(k)} \otimes H^{\otimes n})^\times \rtimes S_n$$

given by

$$X_1 \longmapsto X^{0,1}, \qquad Y_1 \longmapsto Y^{0,1}, \qquad \sigma_i \longmapsto (i, i+1)\mathcal{R}^{i,i+1}.$$

Proof. The two first set of cross relations can obviously be taken as a definition of X_i, Y_i for $i > 1$. That these operators pairwise commute follows from the reflection equation and the Yang-Baxter equation. The last cross relation is nothing but the defining equation (1.2) of $E_H^{(k)}$. □

Let $\widetilde{SL_2(\mathbb{Z})}$ denote the group generated by A, B, Z with relations:

$$A^4 = (AB)^3 = Z, \qquad\qquad (A^2, B) = 1. \tag{4.3}$$

Clearly, Z is central, so this is a central extension,

$$1 \to \mathbb{Z} \to \widetilde{SL_2(\mathbb{Z})} \to SL_2(\mathbb{Z}) \to 1.$$

Proposition 4.4. *The group $\widetilde{SL_2(\mathbb{Z})}$ acts on $B_n(T^2\backslash disc)$ in the following way:*

$$\begin{aligned} A \cdot \sigma_i &= \sigma_i & B \cdot \sigma_i &= \sigma_i \\ A \cdot X_1 &= Y_1 & A \cdot Y_1 &= Y_1 X_1^{-1} Y_1^{-1} \\ B \cdot X_1 &= X_1 & B \cdot Y_1 &= Y_1 X_1^{-1}. \end{aligned}$$

Proposition 4.5. *Let B be an algebra and $(X_B, Y_B) \in B \otimes H$ satisfying equation (1.2) and axioms (2.1) with $k = 1$. Then, so does $(X_B, Y_B X_B^{-1})$ and $(Y, Y_B X_B^{-1} Y_B^{-1})$.*

Proof. Equation (1.2) is exactly one of the defining relation of $B^1_{1,n}$ so that it is satisfied follows from the previous proposition. So we just have to check that $Y_B X_B^{-1}$ and $Y_B X_B^{-1} Y_B^{-1}$ satisfies (2.1) with $k = 1$. This is a direct computation:

$$\begin{aligned} (Y_B X_B^{-1})^{0,12} &= \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} \\ &= \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} Y_B^{0,1} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} \\ &= \mathcal{R}^{2,1} Y_B^{0,2} \mathcal{R}^{1,2} (\mathcal{R}^{1,2})^{-1} (X_B^{0,2})^{-1} \mathcal{R}^{1,2} Y_B^{0,1} \mathcal{R}^{2,1} (\mathcal{R}^{2,1})^{-1} (X_B^{0,1})^{-1} \\ &= \mathcal{R}^{2,1} Y_B^{0,2} (X_B^{0,2})^{-1} \mathcal{R}^{1,2} Y_B^{0,1} (X_B^{0,1})^{-1}, \end{aligned}$$

where at lines 2 and 3 we use the reflection equation and the elliptic commutation relation respectively. The second part is proved by doing the exact same computation replacing Y_B by $Y_B X_B^{-1}$ and X_B by Y_B. □

Corollary 4.6. *There is an action of $\widetilde{SL_2(\mathbb{Z})}$ on $E_H^{(1)}$, uniquely determined by its action on canonical elements X, Y as follows:*

$$A \cdot X = Y, \qquad A \cdot Y = YX^{-1}Y^{-1},$$
$$B \cdot X = X, \qquad B \cdot Y = YX^{-1}.$$

Moreover, the action is compatible with the $\widetilde{SL_2(\mathbb{Z})}$-action on $B_n(T^2 \backslash disc)$,

Proof. It follows from Proposition 4.5 together with the universal property stated in Corollary 3.5. □

5. Relation with the Heisenberg double and quantum Fourier transform

Since $\tilde{H}_0$ is a $H^{[2],coop}$-module algebra, one can form the semi-direct product $\tilde{H} \rtimes H^{[2],coop}$. It is easily checked that $H \otimes 1 \subset H^{[2],coop}$ is a coideal subalgebra, hence the following definition makes sense:

Definition 5.1. *The Heisenberg double D_H is the subalgebra $\tilde{H}_0 \rtimes (H \otimes 1)$.*

Remark 5.2. The standard definition of the Heisenberg double involves H^e and the usual dual, instead of $H^{[2]}$ and the braided dual. However, it is shown in [VV] that these two algebras are isomorphic.

Clearly, the double braiding $\mathcal{R}^{2,1}\mathcal{R}^{1,2}$ satisfies axiom (2.1) with $k = 0$. This is a manifestation of the embedding of the cylinder braid group on n strands into the ordinary braid group on $n+1$ strands. We have:

Theorem 5.3. [Jo] *The canonical element $X \in D_H \otimes H$ together with the image of the double braiding under the inclusion $H \otimes H \to D_H \otimes H$ satisfy the commutation relation* (1.2)*.*

Corollary 5.4. *There exists a canonical map from the elliptic double to the Heisenberg double.*

By construction, this map is the identity on the first $\tilde{H}_0$ component and defined on the second component by the factorization map,

$$\phi : \tilde{H}_0 \to H,$$
$$f \mapsto (f \otimes id)(\mathcal{R}^{2,1}\mathcal{R}^{1,2}).$$

Definition 5.5. *A quasi-triangular Hopf algebra is called factorizable if ϕ is injective.*

Let I_H be the image of ϕ and let D'_H be the subalgebra $\tilde{H} \rtimes (I_H \otimes 1)$ of D_H.

Theorem 5.6. *If H is a factorizable Hopf algebra, then D'_H is isomorphic as an algebra to $E_H^{(0)}$.*

Let G be a reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $U = U_q(\mathfrak{g})$ the corresponding quantum group. Recall (see e.g. [CP, Chap. 9]) that this is a quasi-triangular Hopf algebra[1] over $\mathbb{C}(q)$ for q a variable which, roughly, specialize to the enveloping algebra of $\mathfrak{g}$ at $q = 1$. Denote by $U' = U_q(\mathfrak{g})'$ its ad-locally finite part.

Theorem 5.7 ([BS, RSTS])**.** *U is a factorizable ribbon Hopf algebra, and the image of the factorization map $(U^*) \to U$ is U'.*

[1]This is not quite true since the R-matrix does not belongs to $U_q(\mathfrak{g})^{\otimes 2}$ but only to a certain completion of it, but it is still enough for our purpose

Let $D_q(G)$ be the subalgebra $\tilde{U} \rtimes U'$ of the Heisenberg double of U. It is a deformation of the algebra of differential operators on G. Thanks to the above theorem, $D_q(G)$ is isomorphic to $E_U^{(0)}$ which is itself isomorphic to $E_U^{(1)}$. Altogether this yields the action of $\widetilde{SL_2(\mathbb{Z})}$ on $D_q(G)$.

6. Relation to classical Fourier transform

In this section we show how the Weyl algebra of $\mathfrak{g}$ and the classical Fourier transform can be obtained both directly as the elliptic double of a certain Hopf algebra and via an appropriate degeneration of the elliptic double of the corresponding quantum group. Let $U_\hbar(\mathfrak{g})$ be the "formal" version of the quantum group. This a topological quasi-triangular Hopf algebra over $\mathbb{C}[[\hbar]]$, where $\hbar$ is a formal variable, deforming the enveloping algebra of $\mathfrak{g}$ and whose definition can be found, e.g., in [CP, Chap. 6]. Since directly taking the classical (i.e. $\hbar = 0$) limit of the elliptic commutation relation gives the commutative algebra $S(\mathfrak{g})^{\otimes 2}$ we will have to consider a slightly more complicated degeneration.

Let $S(\mathfrak{g})$ denote the symmetric algebra on $\mathfrak{g}$, equipped with its standard coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$, making it a commutative, cocommutative Hopf algebra. Let $r \in \mathfrak{g}^{\otimes 2}$ denote the quasi-classical limit of the R-matrix of $U_\hbar(\mathfrak{g})$, i.e.:

$$\mathcal{R} = 1 + \hbar r + O(\hbar^2).$$

Then, in a straightforward way, the completion of the symmetric algebra $(\widehat{S}(\mathfrak{g}), \mathcal{R}_0 = \exp(r))$ is a quasi-triangular, factorizable Hopf algebra[2]. Let $t = r + r^{2,1} \in S^2(\mathfrak{g})^{\mathfrak{g}}$ and let C denote the corresponding Casimir element, i.e. $C = m(t)$ where m is the multiplication of $S(\mathfrak{g})$. Then $\nu_0 = \exp(-C/2)$ is a ribbon element. Since $\mathcal{R}_0 \notin S(\mathfrak{g})^{\otimes 2}$, $S(\mathfrak{g})$ is not strictly speaking a ribbon Hopf algebra, but the construction of the elliptic double is still well defined in this situation.

Let $D(\mathfrak{g})$ be the algebra of differential operators on $\mathfrak{g}$, i.e. the Weyl algebra. As a vector space it is $S(\mathfrak{g}^*)^{\otimes 2}$, the two copies of $S(\mathfrak{g}^*)$ are subalgebras and the cross relations are:

$$\forall f, g \in \mathfrak{g}^*, [f \otimes 1, 1 \otimes g] = \langle f, g \rangle$$

where $\langle\ ,\ \rangle$ is the pairing on $\mathfrak{g}^*$ induced by t. The first result of this section is:

Proposition 6.1. *The 0th elliptic double of $(S(\mathfrak{g}), \mathcal{R}_0)$ is isomorphic to the Weyl algebra $D(\mathfrak{g})$ and the action of the generator A of $\widetilde{SL_2(\mathbb{Z})}$ coincides with the classical Fourier transform. That is, on generators $(f, g) \in \mathfrak{g}^* \times \mathfrak{g}^* \subset D(\mathfrak{g})$, we have,*

$$A(f, g) = (-g, f).$$

Proof. Let x, y denote two copies of the canonical element in $\mathfrak{g}^* \otimes \mathfrak{g}$. The restricted dual of $S(\mathfrak{g})$ is $S(\mathfrak{g}^*)$ and the corresponding canonical element is $X = \exp(x)$. Since $S(\mathfrak{g})$ is commutative, equation (2.1) reduces to the standard relation,

$$(\mathrm{id} \otimes \Delta)(X) = X^{0,1} X^{0,2},$$

hence the braided dual and the restricted dual coincide. Likewise, the defining equation of the elliptic double reduces to:

$$(X^{0,1}, Y^{0,2}) = \mathcal{R}_0^{2,1} \mathcal{R}_0^{1,2},$$

where $(a, b) = aba^{-1}b^{-1}$ and $Y = \exp(y)$. Since

$$[x^{0,1}, t^{1,2}] = [y^{0,2}, t^{1,2}] = 0,$$

this equation is equivalent to:

$$[x^{0,1}, y^{0,2}] = t^{1,2}.$$

[2]Here the tensor product is the topological one, i.e. $\widehat{S}(\mathfrak{g})^{\otimes 2} := \widehat{S}(\mathfrak{g} \times \mathfrak{g})$

Applying f and g to the first and second components, respectively, of the above equation gives the defining relations (6) of $D(\mathfrak{g})$.

Since $(S(\mathfrak{g}), \mathcal{R}_0)$ is ribbon, $E^{(0)}_{S(\mathfrak{g})}$ is isomorphic to $E^{(1)}_{S(\mathfrak{g})}$. Pulling back the action of the A generator of $\widetilde{SL_2(\mathbb{Z})}$ through this isomorphism, we find:

$$x \mapsto y \qquad\qquad y \mapsto Y^{-1}(-x + (1 \otimes C))Y$$

It is easily seen that the cross relations of $D(\mathfrak{g})$ implies

$$Y^{-1}xY = x + (1 \otimes C).$$

Hence A map x to y and y to $-x$. □

Let $U_{\hbar^2}(\mathfrak{g})$ be the $\mathbb{C}[[\hbar]]$-Hopf algebra obtained by formally replacing $\hbar$ by $\hbar^2$ in the definition of the product, the coproduct and the R-matrix of $U_\hbar(\mathfrak{g})$. Denote by δ_n the map $(\mathrm{id} - \epsilon)^{\otimes n} \circ \Delta^n$ where ϵ is the counit of $U_{\hbar^2}(\mathfrak{g})$. Denote by $\widehat{U}$ the quantum formal series Hopf algebra (QFSHA) attached to $U_{\hbar^2}(\mathfrak{g})$, i.e. the sub-algebra

$$\widehat{U} = \{x \in U_{\hbar^2}(\mathfrak{g}),\ \delta_n(x) \in \hbar^n U_{\hbar^2}(\mathfrak{g}),\ \forall n \geq 0\}$$

It is known [Dr1, Ga] that $\widehat{U}$ is a flat deformation of $\widehat{S}(\mathfrak{g})$. Hence, choose a $\mathbb{C}[[\hbar]]$-module identification

$$\psi : \widehat{U} \longrightarrow \widehat{S}(\mathfrak{g})[[\hbar]]$$

which is the identity modulo $\hbar$, and let $U \subset \widehat{U}$ be the preimage under ψ of $S(\mathfrak{g})[[\hbar]]$.

Proposition 6.2. *The following holds:*

(a) U is a Hopf algebra.
(b) We have canonical bialgebra isomorphisms:

$$\widehat{U}/(\hbar) \cong \widehat{S}(\mathfrak{g}), \qquad\qquad U/(\hbar) \cong S(\mathfrak{g}).$$

(c) The R-matrix of $U_{\hbar^2}(\mathfrak{g})$ belongs to $\widehat{U}^{\otimes 2}$ and its image in $\widehat{S}(\mathfrak{g})^{\otimes 2}$ is $\mathcal{R}_0$.

One can therefore consider the 0th elliptic double of U. A direct consequence of the above proposition is then:

Corollary 6.3. *The algebra E_U is a flat deformation of the Weyl algebra $D(\mathfrak{g})$, and the $\widetilde{SL_2(\mathbb{Z})}$-action on E_U degenerates to the $\widetilde{SL_2(\mathbb{Z})}$-action on $D(\mathfrak{g})$. In particular, the quantum Fourier transform degenerates to the classical one.*

Proof of Prop. 6.2. All of this can be checked explicitly. A more conceptual argument is as follows: recall that $(\mathfrak{g}, \mu, \delta, r)$ is a quasi-triangular Lie bialgebra, where we denote by μ its bracket and by δ its co-bracket. The quantum group $U_{\hbar^2}(\mathfrak{g})$ is obtained by applying an Etingof–Kazhdan quantization functor [EK] to the $\mathbb{C}[[\hbar]]$-quasi-triangular Lie bialgebra $(\mathfrak{g}[[\hbar]], \mu, \hbar^2\delta, \hbar^2 r)$. On the other hand, $\widehat{U}$ is the quasi-triangular Hopf algebra obtained by applying the same functor to the quasi-triangular Lie bialgebra $(\mathfrak{g}[[\hbar]], \hbar\mu, \hbar\delta, r)$. The QFSHA construction is the lift of the inclusion,

$$(\mathfrak{g}[[\hbar]], \hbar\mu, \hbar\delta, r) \longrightarrow (\mathfrak{g}[[\hbar]], \mu, \hbar^2\delta, \hbar^2 r),$$

given by $x \mapsto \hbar x$ (since $r \in \mathfrak{g}^{\otimes 2}$, its image is indeed $\hbar^2 r$).

One can show that the product, the coproduct and the antipode on $\widehat{U}$ restrict to a well-defined Hopf algebra structure on U. By construction, the reduction modulo $\hbar$ of $\widehat{U}$ is the quantization of the $\mathbb{C}$-quasi-triangular Lie bialgebra,

$$(\mathfrak{g}[[\hbar]], \hbar\mu, \hbar\delta, r)/(\hbar) \cong (\mathfrak{g}, 0, 0, r),$$

which is easily seen to be $(\widehat{S}(\mathfrak{g}), \mathcal{R}_0)$. □

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